MATH 1A - MOCK MIDTERM 1 - SOLUTIONS

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1. *(10 points)* Find the domain of $f(x) = \ln(x) + \sqrt{x^2 - 1}$

We want:

(i) $\boxed{x > 0}$ (because the number under the $\ln$ has to be positive)

(ii) $x^2 - 1 \geq 0$ (the number under the $\sqrt{}$ has to be nonnegative), which is the same as $x^2 \geq 1$, which is the same as $\boxed{x \geq 1 \text{ or } x \leq -1}$

Combining those two facts (draw a picture if necessary), we get that the domain of f is: $\boxed{x \geq 1}$, that is, $\boxed{[1, \infty)}$

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2. *(10 points, 5 points each)* In the following problem, you do **not** have to graph the resulting functions. **BE BRIEF!**

(a) Explain in words how to obtain the graph of $y = 2x^3 - 1$ from the graph of $y = x^3$

1) Stretch the graph of $y = x^3$ vertically by a factor of 2.

2) Shift the resulting graph down one unit.

(b) Explain in words how to obtain the graph of $y = \sin(-2x + 3)$ from the graph of $y = \sin(x)$

1) Shift the graph of $y = \sin(x)$ to the left 3 units

2) Compress the resulting graph horizontally by a factor of 2

3) Flip the resulting graph about the $y-$axis

Note: You can switch 2) and 3), but it's important that 1) comes first!

3. *(10 points)* Find $f^{-1}(x)$, where $f(x) = \ln(2x+3)$

Note: Make sure to write your final answer in terms of x.

1) Let $y = \ln(2x+3)$

2)

$$\begin{aligned} y =& \ln(2x+3) \\ e^y =& e^{\ln(2x+3)} \\ e^y =& 2x+3 \\ 2x+3 =& e^y \\ x =& \frac{e^y - 3}{2} \end{aligned}$$

3) Hence $\boxed{f^{-1}(x) = \frac{e^x - 3}{2}}$

4. *(15 points)* Evaluate $\sin(\tan^{-1}(x))$

Note: Show your steps. You are not just graded on the correct answer, but also on the way you write up your answer.

Here is the bare minimum you'd have to write down:

Let $\theta = \tan^{-1}(x)$, then $\tan(\theta) = x$.

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Then:

$$\sin(\tan^{-1}(x)) = \sin(\theta) = \frac{AC}{BC} \overset{PYTH}{=} \frac{x}{\sqrt{1+x^2}}$$

5. *(40 points, 5 points each)* Evaluate the following limits (or say 'it does not exist'). **Briefly show your work!** :

(a)

$$\begin{aligned}
\lim_{x\to 1}\frac{\sqrt{x+3}-2}{x-1} &= \lim_{x\to 1}\frac{(\sqrt{x+3}-2)(\sqrt{x+3}+2)}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1}\frac{x+3-4}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1}\frac{x-1}{(x-1)(\sqrt{x+3}+2)} \\
&= \lim_{x\to 1}\frac{1}{\sqrt{x+3}+2} \\
&= \frac{1}{\sqrt{4}+2} \\
&= \frac{1}{4}
\end{aligned}$$

(b)

$$\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4} = \lim_{x\to 4}\frac{x(x-4)}{(x-4)(x+1)} = \lim_{x\to 4}\frac{x}{x+1} = \frac{4}{5}$$

(c)

$$\lim_{x\to\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to\infty}\frac{\sqrt{x^2}\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty}\frac{|x|\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty}\frac{x\sqrt{1+\frac{1}{x^2}}}{x} = \lim_{x\to\infty}\sqrt{1+\frac{1}{x^2}} = 1$$

(d) $\lim_{x\to 0}\frac{1}{x}-\frac{1}{|x|}$

$$\lim_{x\to 0^+}\frac{1}{x}-\frac{1}{|x|} = \lim_{x\to 0^+}\frac{1}{x}-\frac{1}{x} = \lim_{x\to 0^+} 0 = 0$$

$$\lim_{x\to 0^-}\frac{1}{x}-\frac{1}{|x|} = \lim_{x\to 0^-}\frac{1}{x}-\left(-\frac{1}{x}\right) = \lim_{x\to 0^-}\frac{2}{x} = -\infty$$

Since the left-hand-side limit and the right-hand-side limits are not equal, the limit **does not exist**

(e) $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right)$

$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$, so $-x^4 \leq \cos\left(\frac{2}{x}\right) \leq x^4$. But $\lim_{x\to 0} -x^4 = 0$ and $\lim_{x\to 0} x^4 = 0$, so by the **squeeze theorem**, $\boxed{\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0}$.

Note: All the steps are important in this problem!

(f)

$$\lim_{x\to\infty} \frac{x^3 - x^2}{x^2 - 1} = \lim_{x\to\infty} \frac{x^3(1 - \frac{1}{x})}{x^2(1 - \frac{1}{x^2})} = \lim_{x\to\infty} \frac{x(1 - \frac{1}{x})}{1 - \frac{1}{x^2}} = \infty \times \frac{1}{1} = \infty$$

(g)

$$\lim_{x\to 3^+} \ln(x^2 - 9) = \ln(0^+) = -\infty$$

(h) $\lim_{x\to\infty} \tan^{-1}(x^2 - x^4)$
(you may use the fact that $\tan^{-1}(-\infty) = -\frac{\pi}{2}$, but justify the fact that you can put the limit inside the $\tan^{-1}$)

First of all, $\lim_{x\to\infty} x^2 - x^4 = \lim_{x\to\infty} x^4(\frac{1}{x^2} - 1) = \infty(-1) = -\infty$.

Now by **CONTINUITY** of $\tan^{-1}(x)$,

$$\lim_{x\to\infty} \tan^{-1}(x^2 - x^4) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

6. *(15 points)* Show that the equation $\cos(x) = x$ has at least one solution.
Show your work: You will be graded not only on the correct answer, but also on the way you write up your answer

Let $f(x) = \cos(x) - x$.

Then $f(0) = \cos(0) - 0 = 1 > 0$ and $f(\pi) = \cos(\pi) - \pi = -1 - \pi < 0$.

But since f is **continuous** on $[0, \pi]$, by **the intermediate value theorem**, f has at least one zero on $(0, \pi)$, hence at least one zero on $\mathbb{R}$, and hence $\cos(x) = x$ has at least one solution.

Note: The main thing I will be looking at are the words 'continuous' and 'intermediate value theorem' and the fact that $f(0) > 0$ and $f(\pi) < 0$.

Bonus 1 *(5 points)* Which function has the property that it is both even and odd? Prove that your answer is correct!

The answer is the **zero function**, $f(x) = 0$. Here's why:

Suppose f is even and odd. Since f is even, $f(-x) = f(x)$. But since f is odd, $f(-x) = -f(x)$. Subtracting the second equation from the first, we get: $f(-x) - f(-x) = f(x) - (-f(x))$, so $0 = 2f(x)$, so $\boxed{f(x) = 0}$.

BOOM, we're done!

Bonus 2 *(5 points)* Given a one-to-one function f, find the inverse of the function $g(x) = f(cx + d)$ (where c, d are nonzero real numbers).

Hint: Think of this in terms of actions. In order to obtain $g(x)$ from x, what do you do first? What do you do second? What do you do last? Now think about how you can 'undo' g.

The answer is $\boxed{g^{-1}(x) = \frac{f^{-1}(x)-d}{c}}$.

This is because in order to obtain g from x, first you multiply x by c, then you add d, and finally you apply f. So if we want to undo this, first we undo f (by applying f^{-1}), then we undo the addition (by subtracting d), and finally we undo the multiplication (by dividing by c)